

THE COMPUTATIONAL CONTENT OF THE LOEB MEASURE

SAM SANDERS

ABSTRACT. The *Loeb measure* is one of the cornerstones of *Nonstandard Analysis*. The traditional development of the Loeb measure makes use of *saturation* and *external sets*. Inspired by [13], we give meaning to special cases of the Loeb measure in the weak fragment \mathbf{P} of Nelson’s *internal set theory* from [1]. Perhaps surprisingly, our definition of the Loeb measure has computational content in the sense of the ‘term extraction’ framework from [1]

1. INTRODUCTION

The *Loeb measure* ([3, 14]) is one of the cornerstones of Robinson’s *Nonstandard Analysis* (NSA for short; see [7]). The traditional development of the Loeb measure in NSA makes use of *saturation* and *external sets*. A special case of the Loeb measure is introduced in a weak fragment of NSA in [13] using external sets, but without the use of *saturation*. The definition of measure from Reverse Mathematics (RM for short see [12] for an overview) is used.

In this paper, we similarly introduce a special case of the Loeb measure, but in the weak fragment \mathbf{P} of Nelson’s *internal set theory* (IST for short; see [6]) from [1]. We show that our definition of the Loeb measure has computational content in the sense of the framework from [1]. In particular, we show that our definition of the Loeb measure falls inside the scope of the ‘term extraction theorem’ of the system \mathbf{P} as in Corollary 2.5 below.

We first introduce Nelson’s internal set theory in Section 2.1 and a fragment called \mathbf{P} based on Gödel’s system \mathbf{T} in Section 2.2. The development of the Loeb measure in \mathbf{P} takes place in Section 3.

2. INTERNAL SET THEORY AND ITS FRAGMENT \mathbf{P}

In this section, we discuss Nelson’s *internal set theory*, first introduced in [6], and its fragment \mathbf{P} from [1]. The latter fragment is essential to our enterprise, especially Corollary 2.5 below.

2.1. Internal set theory 101. In Nelson’s *syntactic* approach to Nonstandard Analysis ([6]), as opposed to Robinson’s semantic one ([7]), a new predicate ‘ $\text{st}(x)$ ’, read as ‘ x is standard’ is added to the language of \mathbf{ZFC} , the usual foundation of mathematics. The notations $(\forall^{\text{st}}x)$ and $(\exists^{\text{st}}y)$ are short for $(\forall x)(\text{st}(x) \rightarrow \dots)$ and $(\exists y)(\text{st}(y) \wedge \dots)$. A formula is called *internal* if it does not involve ‘ st ’, and

MUNICH CENTER FOR MATHEMATICAL PHILOSOPHY, LMU MUNICH, GERMANY
E-mail address: `sasander@me.com`.

external otherwise. The three external axioms *Idealisation*, *Standard Part*, and *Transfer* govern the new predicate ‘st’; They are respectively defined¹ as:

- (I) $(\forall^{\text{st fin}} x)(\exists y)(\forall z \in x)\varphi(z, y) \rightarrow (\exists y)(\forall^{\text{st}} x)\varphi(x, y)$, for internal φ with any (possibly nonstandard) parameters.
- (S) $(\forall^{\text{st}} x)(\exists^{\text{st}} y)(\forall^{\text{st}} z)((z \in x \wedge \varphi(z)) \leftrightarrow z \in y)$, for any φ .
- (T) $(\forall^{\text{st}} t)[(\forall^{\text{st}} x)\varphi(x, t) \rightarrow (\forall x)\varphi(x, t)]$, where $\varphi(x, t)$ is internal, and only has free variables t, x .

The system IST is (the internal system) ZFC extended with the aforementioned external axioms; The former is a conservative extension of ZFC for the internal language, as proved in [6].

In [1], the authors study Gödel’s system T extended with special cases of the external axioms of IST. In particular, they consider the systems H and P, introduced in the next section, which are conservative extensions of the (internal) logical systems E-HA^ω and E-PA^ω , respectively *Heyting and Peano arithmetic in all finite types and the axiom of extensionality*. We refer to [4, §3.3] for the exact definitions of the (mainstream in mathematical logic) systems E-HA^ω and E-PA^ω . Furthermore, $\text{E-PA}^{\omega*}$ and $\text{E-HA}^{\omega*}$ are the definitional extensions of E-PA^ω and E-HA^ω with types for finite sequences, as in [1, §2]. For the former systems, we require some notation.

2.1. Notation (Finite sequences). The systems $\text{E-PA}^{\omega*}$ and $\text{E-HA}^{\omega*}$ have a dedicated type for ‘finite sequences of objects of type ρ ’, namely ρ^* . Since the usual coding of pairs of numbers goes through in both, we shall not always distinguish between 0 and 0^* . Similarly, we do not always distinguish between ‘ s^ρ ’ and ‘ $\langle s^\rho \rangle$ ’, where the former is ‘the object s of type ρ ’, and the latter is ‘the sequence of type ρ^* with only element s^ρ ’. The empty sequence for the type ρ^* is denoted by ‘ $\langle \rangle_\rho$ ’, usually with the typing omitted. Furthermore, we denote by ‘ $|s| = n$ ’ the length of the finite sequence $s^{\rho^*} = \langle s_0^\rho, s_1^\rho, \dots, s_{n-1}^\rho \rangle$, where $|\langle \rangle| = 0$, i.e. the empty sequence has length zero. For sequences s^{ρ^*}, t^{ρ^*} , we denote by ‘ $s * t$ ’ the concatenation of s and t , i.e. $(s * t)(i) = s(i)$ for $i < |s|$ and $(s * t)(j) = t(|s| - j)$ for $|s| \leq j < |s| + |t|$. For a sequence s^{ρ^*} , we define $\bar{s}N := \langle s(0), s(1), \dots, s(N) \rangle$ for $N^0 < |s|$. For a sequence $\alpha^{0 \rightarrow \rho}$, we also write $\bar{\alpha}N = \langle \alpha(0), \alpha(1), \dots, \alpha(N) \rangle$ for any N^0 . By way of shorthand, $q^\rho \in Q^{\rho^*}$ abbreviates $(\exists i < |Q|)(Q(i) =_\rho q)$. Finally, we shall use $\underline{x}, \underline{y}, \underline{t}, \dots$ as short for tuples $x_0^{\sigma_0}, \dots, x_k^{\sigma_k}$ of possibly different type σ_i .

2.2. The classical system P. In this section, we introduce the system P, a conservative extension of E-PA^ω with fragments of Nelson’s IST.

To this end, we first introduce the base system $\text{E-PA}_{\text{st}}^{\omega*}$. We use the same definition as [1, Def. 6.1], where $\text{E-PA}^{\omega*}$ is the definitional extension of E-PA^ω with types for finite sequences as in [1, §2]. The set \mathcal{T}^* is defined as the collection of all the terms in the language of $\text{E-PA}^{\omega*}$.

2.2. Definition. The system $\text{E-PA}_{\text{st}}^{\omega*}$ is defined as $\text{E-PA}^{\omega*} + \mathcal{T}_{\text{st}}^* + \text{IA}^{\text{st}}$, where $\mathcal{T}_{\text{st}}^*$ consists of the following axiom schemas.

- (1) The schema² $\text{st}(x) \wedge x = y \rightarrow \text{st}(y)$,

¹The superscript ‘fin’ in (I) means that x is finite, i.e. its number of elements are bounded by a natural number.

²The language of $\text{E-PA}_{\text{st}}^{\omega*}$ contains a symbol st_σ for each finite type σ , but the subscript is essentially always omitted. Hence $\mathcal{T}_{\text{st}}^*$ is an *axiom schema* and not an axiom.

- (2) The schema providing for each closed³ term $t \in \mathcal{T}^*$ the axiom $\text{st}(t)$.
- (3) The schema $\text{st}(f) \wedge \text{st}(x) \rightarrow \text{st}(f(x))$.

The external induction axiom IA^{st} is as follows.

$$\Phi(0) \wedge (\forall^{\text{st}} n^0)(\Phi(n) \rightarrow \Phi(n+1)) \rightarrow (\forall^{\text{st}} n^0)\Phi(n). \quad (\text{IA}^{\text{st}})$$

Secondly, we introduce some essential fragments of IST studied in [1].

2.3. Definition. [External axioms of P]

- (1) HAC_{int} : For any internal formula φ , we have

$$(\forall^{\text{st}} x^\rho)(\exists^{\text{st}} y^\tau)\varphi(x, y) \rightarrow (\exists^{\text{st}} F^{\rho \rightarrow \tau^*})(\forall^{\text{st}} x^\rho)(\exists y^\tau \in F(x))\varphi(x, y), \quad (2.1)$$

- (2) I : For any internal formula φ , we have

$$(\forall^{\text{st}} x^{\sigma^*})(\exists y^\tau)(\forall z^\sigma \in x)\varphi(z, y) \rightarrow (\exists y^\tau)(\forall^{\text{st}} x^\sigma)\varphi(x, y),$$

- (3) The system P is $\text{E-PA}_{\text{st}}^{\omega*} + \text{I} + \text{HAC}_{\text{int}}$.

Note that I and HAC_{int} are fragments of Nelson's axioms *Idealisation* and *Standard part*. By definition, F in (2.1) only provides a *finite sequence* of witnesses to $(\exists^{\text{st}} y)$, explaining its name *Herbrandized Axiom of Choice*.

The system P is connected to E-PA^ω by the following theorem. Here, the superscript ' S_{st} ' is the syntactic translation defined in [1, Def. 7.1].

2.4. Theorem. *Let $\Phi(\underline{a})$ be a formula in the language of $\text{E-PA}_{\text{st}}^{\omega*}$ and suppose $\Phi(\underline{a})^{S_{\text{st}}} \equiv \forall^{\text{st}} \underline{x} \exists^{\text{st}} \underline{y} \varphi(\underline{x}, \underline{y}, \underline{a})$. If Δ_{intern} is a collection of internal formulas and*

$$\text{P} + \Delta_{\text{intern}} \vdash \Phi(\underline{a}), \quad (2.2)$$

then one can extract from the proof a sequence of closed⁴ terms t in \mathcal{T}^ such that*

$$\text{E-PA}^{\omega*} + \Delta_{\text{intern}} \vdash \forall \underline{x} \exists \underline{y} \in \underline{t}(\underline{x}) \varphi(\underline{x}, \underline{y}, \underline{a}). \quad (2.3)$$

Proof. Immediate by [1, Theorem 7.7]. \square

The proofs of the soundness theorems in [1, §5-7] provide an algorithm \mathcal{A} to obtain the term t from the theorem. In particular, these terms can be 'read off' from the nonstandard proofs.

In light of the results in [10], the following corollary (which is not present in [1]) is essential to our results. Indeed, the following corollary expresses that we may obtain effective results as in (2.5) from any theorem of Nonstandard Analysis which has the same form as in (2.4). It was shown in [8–10] that the scope of this corollary includes the Big Five systems of Reverse Mathematics and the associated 'zoo' ([2]).

2.5. Corollary. *If Δ_{intern} is a collection of internal formulas and ψ is internal, and*

$$\text{P} + \Delta_{\text{intern}} \vdash (\forall^{\text{st}} \underline{x})(\exists^{\text{st}} \underline{y})\psi(\underline{x}, \underline{y}, \underline{a}), \quad (2.4)$$

then one can extract from the proof a sequence of closed⁴ terms t in \mathcal{T}^ such that*

$$\text{E-PA}^{\omega*} + \text{QF-AC}^{1,0} + \Delta_{\text{intern}} \vdash (\forall \underline{x})(\exists \underline{y} \in \underline{t}(\underline{x}))\psi(\underline{x}, \underline{y}, \underline{a}). \quad (2.5)$$

³A term is called *closed* in [1] (and in this paper) if all variables are bound via lambda abstraction. Thus, if $\underline{x}, \underline{y}$ are the only variables occurring in the term t , the term $(\lambda \underline{x})(\lambda \underline{y})t(\underline{x}, \underline{y})$ is closed while $(\lambda \underline{x})t(\underline{x}, \underline{y})$ is not. The second axiom in Definition 2.2 thus expresses that $\text{st}_\tau((\lambda \underline{x})(\lambda \underline{y})t(\underline{x}, \underline{y}))$ if $(\lambda \underline{x})(\lambda \underline{y})t(\underline{x}, \underline{y})$ is of type τ . We usually omit lambda abstraction for brevity.

⁴Recall the definition of closed terms from [1] as sketched in Footnote 3.

Proof. Clearly, if for internal ψ and $\Phi(\underline{a}) \equiv (\forall^{\text{st}} \underline{x})(\exists^{\text{st}} \underline{y})\psi(x, y, a)$, we have $[\Phi(\underline{a})]^{S_{\text{st}}} \equiv \Phi(\underline{a})$, then the corollary follows immediately from the theorem. A tedious but straightforward verification using the clauses (i)-(v) in [1, Def. 7.1] establishes that indeed $\Phi(\underline{a})^{S_{\text{st}}} \equiv \Phi(\underline{a})$. \square

For the rest of this paper, the notion ‘normal form’ shall refer to a formula as in (2.4), i.e. of the form $(\forall^{\text{st}} x)(\exists^{\text{st}} y)\varphi(x, y)$ for φ internal.

Finally, the previous theorems do not really depend on the presence of full Peano arithmetic. We shall study the following subsystems.

2.6. Definition.

- (1) Let E-PRA^ω be the system defined in [5, §2] and let $\text{E-PRA}^{\omega*}$ be its definitional extension with types for finite sequences as in [1, §2].
- (2) $(\text{QF-AC}^{\rho, \tau})$ For every quantifier-free internal formula $\varphi(x, y)$, we have

$$(\forall x^\rho)(\exists y^\tau)\varphi(x, y) \rightarrow (\exists F^{\rho \rightarrow \tau})(\forall x^\rho)\varphi(x, F(x)) \quad (2.6)$$

- (3) The system RCA_0^ω is $\text{E-PRA}^\omega + \text{QF-AC}^{1,0}$.

The system RCA_0^ω is the ‘base theory of higher-order Reverse Mathematics’ as introduced in [5, §2]. We permit ourselves a slight abuse of notation by also referring to the system $\text{E-PRA}^{\omega*} + \text{QF-AC}^{1,0}$ as RCA_0^ω .

2.7. Corollary. *The previous theorem and corollary go through for \mathbf{P} and $\text{E-PA}^{\omega*}$ replaced by $\mathbf{P}_0 \equiv \text{E-PRA}^{\omega*} + \mathcal{T}_{\text{st}}^* + \text{HAC}_{\text{int}} + \mathbf{I} + \text{QF-AC}^{1,0}$ and RCA_0^ω .*

Proof. The proof of [1, Theorem 7.7] goes through for any fragment of $\text{E-PA}^{\omega*}$ which includes EFA , sometimes also called $\mathbf{I}\Delta_0 + \text{EXP}$. In particular, the exponential function is (all what is) required to ‘easily’ manipulate finite sequences. \square

2.3. Notations. We mostly use the notations from [1], some of which we repeat.

2.8. Remark (Notations). We write $(\forall^{\text{st}} x^\tau)\Phi(x^\tau)$ and $(\exists^{\text{st}} x^\sigma)\Psi(x^\sigma)$ as short for $(\forall x^\tau)[\text{st}(x^\tau) \rightarrow \Phi(x^\tau)]$ and $(\exists x^\sigma)[\text{st}(x^\sigma) \wedge \Psi(x^\sigma)]$. A formula A is ‘internal’ if it does not involve st ; the formula A^{st} is defined from A by appending ‘ st ’ to all quantifiers (except bounded number quantifiers).

Secondly, we will use the usual notations for rational and real numbers and functions as introduced in [5, p. 288-289] (and [12, I.8.1] for the former).

2.9. Definition (Real numbers etc.). A (standard) real number x is a (standard) fast-converging Cauchy sequence $q_{(\cdot)}^1$, i.e. $(\forall n^0, i^0)(|q_n - q_{n+i}| <_0 \frac{1}{2^n})$. We freely make use of Kohlenbach’s ‘hat function’ from [5, p. 289] to guarantee that every sequence f^1 can be viewed as a real. We also use the notation $[x](k) := q_k$ for the k -th approximation of real numbers. Two reals x, y represented by $q_{(\cdot)}$ and $r_{(\cdot)}$ are *equal*, denoted $x =_{\mathbb{R}} y$, if $(\forall n)(|q_n - r_n| \leq \frac{1}{2^n})$. Inequality $<_{\mathbb{R}}$ is defined similarly. We also write $x \approx y$ if $(\forall^{\text{st}} n)(|q_n - r_n| \leq \frac{1}{2^n})$ and $x \gg y$ if $x >_{\mathbb{R}} y \wedge x \not\approx y$. Functions $F : \mathbb{R} \rightarrow \mathbb{R}$ are represented by $\Phi^{1 \rightarrow 1}$ such that

$$(\forall x, y)(x =_{\mathbb{R}} y \rightarrow \Phi(x) =_{\mathbb{R}} \Phi(y)), \quad (\text{RE})$$

i.e. equal reals are mapped to equal reals. Finally, sets are denoted X^1, Y^1, Z^1, \dots and are given by their characteristic functions f_X^1 , i.e. $(\forall x^0)[x \in X \leftrightarrow f_X(x) = 1]$, where f_X^1 is assumed to be binary.

Thirdly, we use the usual extensional notion of equality.

2.10. Remark (Equality). Equality between natural numbers ‘ $=_0$ ’ is a primitive. Equality ‘ $=_\tau$ ’ for type τ -objects x, y is then defined as follows:

$$[x =_\tau y] \equiv (\forall z_1^{\tau_1} \dots z_k^{\tau_k}) [xz_1 \dots z_k =_0 yz_1 \dots z_k] \quad (2.7)$$

if the type τ is composed as $\tau \equiv (\tau_1 \rightarrow \dots \rightarrow \tau_k \rightarrow 0)$. In the spirit of Nonstandard Analysis, we define ‘approximate equality \approx_τ ’ as follows:

$$[x \approx_\tau y] \equiv (\forall^{\text{st}} z_1^{\tau_1} \dots z_k^{\tau_k}) [xz_1 \dots z_k =_0 yz_1 \dots z_k] \quad (2.8)$$

with the type τ as above. The system \mathbf{P} includes the *axiom of extensionality*:

$$(\forall x^\rho, y^\rho, \varphi^{\rho \rightarrow \tau}) [x =_\rho y \rightarrow \varphi(x) =_\tau \varphi(y)]. \quad (\text{E})$$

However, as noted in [1, p. 1973], the so-called axiom of *standard* extensionality $(\text{E})^{\text{st}}$ is problematic and cannot be included in \mathbf{P} .

3. THE LOEB MEASURE IN \mathbf{P}

In this section, we discuss the Loeb measure in the context of internal set theory IST , and possible computational aspects thereof. This development takes place in the system \mathbf{P} from the previous section. We assume basic familiarity with *Reverse Mathematics* (RM for short) and we refer to [11, 12] for an overview of the latter program, the definition of the ‘Big Five’, and the base theory RCA_0 in particular.

First of all, the usual definition of the Loeb measure L_M (See Definition 3.2 below) makes use of *external* sets, and therefore seems meaningless in IST . Nonetheless, we shall see that one can give meaning to the formula ‘ $L_M(A) = 0$ ’ inside \mathbf{P} , even though ‘ $L_M(A)$ ’ strictly speaking does not exist. This is reminiscent of the situation of measure theory in Reverse Mathematics (See e.g. [15, 16]), where the Lebesgue measure is defined as follows in [12, X.1.2-3].

3.1. Definition. [Lebesgue measure λ] For $\|g\| := \int_0^1 g(x)dx$, we define

$$\lambda(U) := \sup\{\|g\| : g \in C([0, 1]) \wedge 0 \leq g \leq 1 \wedge (\forall x \in [0, 1] \setminus U)(g(x) = 0)\}.$$

Of course, this supremum does not necessarily exist in weak systems such as the base theory RCA_0 of RM, but the formula ‘ $\lambda(U) =_{\mathbb{R}} 0$ ’ defined as follows makes perfect sense in weak systems such as RCA_0 :

$$[\lambda(U) =_{\mathbb{R}} 0] \equiv (\forall g \in C([0, 1])) [0 \leq g \leq 1 \wedge (\forall x \in [0, 1] \setminus U)g(x) = 0 \rightarrow \|g\| = 0].$$

Note that the existence of the Lebesgue measure for open sets is actually equivalent to ACA_0 by [12, p. 391]. We conclude that while the Lebesgue measure λ may not exist in weak systems of RM, the formula $\lambda(U) =_{\mathbb{R}} 0$ always is meaningful.

Below, we show that a similar trick can be used to give meaning to the Loeb measure in IST . Thus, fix nonstandard M and consider the grid $\mathcal{G}_M = \{\frac{i}{2^M} : 0 \leq i \leq 2^M\}$ on $[0, 1]$. The usual definition of the Loeb measure from [13] is as follows.

3.2. Definition. [Loeb measure] The Loeb measure of a set $B \subseteq \mathcal{G}_M$ is $L_M(B) := \text{st}(L_M^*(B))$, where $L_M^*(B) := \frac{|B|}{2^M}$. The Loeb measure of a set $A \subseteq [0, 1]$ is defined as follows.

$$\text{st}_M^{-1}(A) := \{b \in \mathcal{G}_M : (\exists^{\text{st}} a^1 \in A)(a \approx b)\}. \quad (3.1)$$

$$C \subset_{\text{al}} D := (\forall E)(E \subset (C \setminus D) \rightarrow L_M^*(E) \approx 0) \quad (C, D \subseteq \mathcal{G}_M).$$

$$L_M(A) := \sup\{L_M(B) : B \subseteq \mathcal{G}_M \wedge B \subset_{\text{al}} \text{st}_M^{-1}(A)\}. \quad (3.2)$$

$$L_M^*(A) := \sup\{L_M^*(B) : B \subseteq \mathcal{G}_M \wedge B \subset_{\text{al}} \text{st}_M^{-1}(A)\}. \quad (3.3)$$

Note that $L_M(A) = \text{st}(L_M^*(A))$ since $L_M(A) \approx L_M^*(A)$. We introduced (3.3) as the IST axiom *Standard Part* is non-constructive, while the standard part map is external, and hence does not exist in IST.

Thirdly, the set $\text{st}_M^{-1}(A)$ as in (3.1) is external, and hence it seems the Loeb measure $L_M(A)$ as in (3.2) cannot be defined in IST. Nonetheless, the formula ' $L_M^*(A) \approx 0$ ' (or equivalently ' $L_M(A) = 0$ ') does make sense in IST, as follows:

$$\begin{aligned} L_M^*(A) \approx 0 &\equiv (\forall B \subseteq \mathcal{G}_M) [B \subset_{\text{al}} \text{st}_M^{-1}(A) \rightarrow L_M^*(B) \approx 0] \\ &\equiv (\forall B \subseteq \mathcal{G}_M) [(\forall E)(E \subset (B \setminus \text{st}_M^{-1}(A)) \rightarrow L_M^*(E) \approx 0) \rightarrow L_M^*(B) \approx 0] \\ &\equiv (\forall B \subseteq \mathcal{G}_M) \left[[(\forall E)((\forall e \in E)(e \in B \wedge (\forall^{\text{st}} a \in A)(a \not\approx e)) \rightarrow L_M^*(E) \approx 0)] \rightarrow L_M^*(B) \approx 0 \right]. \end{aligned}$$

Note that the final formula is a formula of IST (and can be expressed in far weaker systems such as P). The formula ' $a \in A$ ' can be replaced by any formula $\Phi(a)$, and we can thus give meaning to the formula $L_M^*(\{a : \Phi(a)\}) \approx 0$. We can now say that a property Φ 'holds *almost everywhere* in $[0, 1]$ ' if $L_M^*(\{a \in [0, 1] : \neg \Phi(a)\}) \approx 0$.

Fourth, recall that we can obtain computational information from formulas of the form $(\forall^{\text{st}} x)(\exists^{\text{st}} y)\varphi(x, y)$, where φ is internal by Corollary 2.5. We refer to such formulas as 'normal forms'. Let us now formulate a normal form for the formula ' $L_M^*(A) \approx 0$ '. A normal form for $(\forall e \in E)(e \in B \wedge (\forall^{\text{st}} a \in A)(a \not\approx e))$ is as follows:

$$\begin{aligned} &(\forall e \in E)(e \in B \wedge (\forall^{\text{st}} a \in A)(a \not\approx e)) \\ &\equiv (\forall^{\text{st}} a \in A)(\forall e \in E)(\exists^{\text{st}} k^0)(e \in B \wedge (|a - e| > \tfrac{1}{k})) \\ &\equiv (\forall^{\text{st}} a \in A)(\exists^{\text{st}} K^{0*})(\forall e \in E)(\exists k^0 \in K)(e \in B \wedge (|a - e| > \tfrac{1}{k})) \quad (3.4) \\ &\equiv (\forall^{\text{st}} a \in A)(\exists^{\text{st}} l^0)[(\forall e \in E)(e \in B \wedge (|a - e| > \tfrac{1}{l}))], \quad (3.5) \end{aligned}$$

where (3.4) follows from applying idealisation **I**, and (3.5) follows from defining $l := \max_{i < |K|} K(i)$ in (3.4). Let $A_0(a, E, B, l)$ be the formula in square brackets in (3.5), and note that ' $L_M^*(A) \approx 0$ ' is:

$$\begin{aligned} &(\forall B \subseteq \mathcal{G}_M) \left[[(\forall E)((\forall e \in E)(e \in B \wedge (\forall^{\text{st}} a \in A)(a \not\approx e)) \rightarrow L_M^*(E) \approx 0)] \rightarrow L_M^*(B) \approx 0 \right]. \\ &\equiv (\forall B \subseteq \mathcal{G}_M) \left[[(\forall E)((\forall^{\text{st}} a \in A)(\exists^{\text{st}} l) A_0(a, E, B, l) \rightarrow (\forall^{\text{st}} k) |L_M^*(E)| \leq \tfrac{1}{k})] \rightarrow L_M^*(B) \approx 0 \right]. \\ &\equiv (\forall B \subseteq \mathcal{G}_M) \left[[(\forall^{\text{st}} k, g)(\forall E)((\forall^{\text{st}} a \in A) A_0(a, E, B, g(a)) \rightarrow |L_M^*(E)| \leq \tfrac{1}{k})] \rightarrow L_M^*(B) \approx 0 \right]. \\ &\equiv (\forall B \subseteq \mathcal{G}_M) \left[[(\forall^{\text{st}} k, g)(\forall E)(\exists^{\text{st}} a \in A)(A_0(a, E, B, g(a)) \rightarrow |L_M^*(E)| \leq \tfrac{1}{k})] \rightarrow L_M^*(B) \approx 0 \right]. \\ &\equiv (\forall B \subseteq \mathcal{G}_M) \left[[(\forall^{\text{st}} k, g)(\exists^{\text{st}} b^{1*}) \underline{(\forall E)(\exists a \in b)(a \in A \wedge A_0(a, E, B, g(a)) \rightarrow |L_M^*(E)| \leq \tfrac{1}{k})}] \rightarrow L_M^*(B) \approx 0 \right], \end{aligned}$$

and let $B_0(B, k, g, b, A)$ be the underlined formula. Hence, ' $L_M^*(A) \approx 0$ ' becomes:

$$\begin{aligned} & (\forall B \subseteq \mathcal{G}_M) \left[[(\forall^{\text{st}} k, g)(\exists^{\text{st}} b^{1^*}) B_0(B, k, g, b, A)] \rightarrow L_M^*(B) \approx 0 \right], \\ & \equiv (\forall^{\text{st}} k', h)(\forall B \subseteq \mathcal{G}_M) \left[[(\forall^{\text{st}} k, g) B_0(B, k, g, h(k, g), A)] \rightarrow |L_M^*(B)| \leq \frac{1}{k'} \right]. \\ & \equiv (\forall^{\text{st}} k', h)(\forall B \subseteq \mathcal{G}_M)(\exists^{\text{st}} k, g) \left[[B_0(B, k, g, h(k, g), A)] \rightarrow |L_M^*(B)| \leq \frac{1}{k'} \right]. \\ & \equiv (\forall^{\text{st}} k', h)(\exists^{\text{st}} w)(\forall B \subseteq \mathcal{G}_M)(\exists k, g \in w) \left[[B_0(B, k, g, h(k, g), A)] \rightarrow |L_M^*(B)| \leq \frac{1}{k'} \right]. \end{aligned}$$

Thus, the final formula is a normal form of ' $L_M^*(A) \approx 0$ '.

Fifth, the formula ' $L_M^*(A) \approx 0$ ' involves a nonstandard number M , and one will usually encounter the latter formula somewhere in the scope of the quantifier $(\forall M^0)(\neg \text{st}(M) \rightarrow \dots)$. Such a nonstandard quantifier place nicely with our normal forms, not just for numbers, but for any finite type ρ .

3.3. Theorem. *For internal φ , the formula*

$$(\forall M^\rho) [\neg \text{st}_\rho(M) \rightarrow (\forall^{\text{st}} x)(\exists^{\text{st}} y) \varphi(x, y, M)], \quad (3.6)$$

is equivalent to a normal form.

Proof. First of all, (3.6) is equivalent to the following by Definition 2.2:

$$(\forall M^\rho) [(\forall^{\text{st}} r^\rho)(M \neq_\rho k) \rightarrow (\forall^{\text{st}} x)(\exists^{\text{st}} y) \varphi(x, y, M)],$$

where ' $x \neq_\rho y$ ' is an internal formula. Pushing outside the standard quantifiers as far as possible, we obtain

$$(\forall^{\text{st}} x)(\forall M^\rho)(\exists^{\text{st}} r, y) [(M \neq_\rho r \rightarrow \varphi(x, y, M)],$$

and applying idealisation I, we obtain the following normal form:

$$(\forall^{\text{st}} x)(\exists^{\text{st}} w)(\forall M^\rho)(\exists r, y \in w) [(M \neq r \rightarrow \varphi(x, y, M)].$$

□

By the previous theorem, one can partition a space in infinitesimal pieces $\frac{1}{M}$ for nonstandard M , and the associated quantifier.

Sixth, it is a natural question which sets A can be studied using the above normal form for $L_M^*(A) \approx 0$. As it turns out, in the definition of $L_M^*(A) \approx 0$, one can replace ' $a \in A$ ' by any normal form $(\forall^{\text{st}} z)(\exists^{\text{st}} w) \varphi(z, w, a)$, and the resulting modification of $L_M^*(A) \approx 0$ remains a normal form. In particular, (3.5) becomes the following normal form with this replacement:

$$\begin{aligned} & (\forall^{\text{st}} a \in A)(\exists^{\text{st}} l^0) [(\forall e \in E)(e \in B \wedge (|a - e| > \frac{1}{l}))] \\ & \equiv (\forall^{\text{st}} a) [(\forall^{\text{st}} z)(\exists^{\text{st}} a) \varphi(z, w, a) \rightarrow (\exists^{\text{st}} l^0) [(\forall e \in E)(e \in B \wedge (|a - e| > \frac{1}{l}))]] \\ & \equiv (\forall^{\text{st}} a, g) [(\forall^{\text{st}} z) \varphi(z, g(z), a) \rightarrow (\exists^{\text{st}} l^0) [(\forall e \in E)(e \in B \wedge (|a - e| > \frac{1}{l}))]] \\ & \equiv (\forall^{\text{st}} a, g)(\exists^{\text{st}} z, l^0) [\varphi(z, g(z), a) \rightarrow [(\forall e \in E)(e \in B \wedge (|a - e| > \frac{1}{l}))]]. \end{aligned}$$

One then obtains a normal form for $L_M^*(A) \approx 0$ in exactly the same way as for (3.5). By way of example, we may take ' $a \in A$ ' to be 'the function f is nonstandard continuous at a ', and ' $L_M^*(A) \approx 0$ ' still has a normal form. The same holds for formulas of the form $(\exists^{\text{st}} u)(\forall^{\text{st}} z)(\exists^{\text{st}} w) \varphi(u, z, w, a)$ by the previous, and thus also for *negations* of normal forms.

Seventh, we consider the following alternative definition of the Loeb measure.

3.4. Definition. [Second Loeb measure] The Loeb measure of a set $B \subseteq \mathcal{G}_M$ is $L_M(B) := \text{st}(L_M^*(B))$, where $L_M^*(B) := \frac{|B|}{2^M}$. The *second* Loeb measure of a set $A \subseteq [0, 1]$ is defined as follows.

$$\text{st}_{M,2}^{-1}(A) := \{b \in \mathcal{G}_M : (\exists^{\text{st}} a^1, c^1 \in \mathbb{R})(a \lesssim b \lesssim c) \wedge (\forall^{\text{st}} e^1 \in \mathbb{R})(e \in [a, c] \rightarrow e \in A)\}.$$

$$C \subset_{\text{al}} D := (\forall E)(E \subset (C \setminus D) \rightarrow L_M^*(E) \approx 0) \quad (C, D \subseteq \mathcal{G}_M).$$

$$L_{M,2}(A) := \sup\{L_M(B) : B \subseteq \mathcal{G}_M \wedge B \subset_{\text{al}} \text{st}_{M,2}^{-1}(A)\}. \quad (3.7)$$

$$L_{M,2}^*(A) := \sup\{L_M^*(B) : B \subseteq \mathcal{G}_M \wedge B \subset_{\text{al}} \text{st}_{M,2}^{-1}(A)\}. \quad (3.8)$$

Note that the formula ‘ $x \notin \text{st}_{M,2}^{-1}(A)$ ’ has a normal form similar to that of ‘ $\text{st}_M^{-1}(A)$ ’, hence the formula $L_{M,2}^*(A) \approx 0$ also has normal form.

In conclusion, we cannot define the Loeb measure $L_M(A)$ in IST, but we can give meaning to formula ‘ $L_M(A) = 0$ ’ (and any other (in)equality in the same way). Furthermore, such formulas have normal forms (and therefore carry numerical information), even if we quantify over the nonstandard number M .

4. BIBLIOGRAPHY

- [1] Benno van den Berg, Eyvind Briseid, and Pavol Safarik, *A functional interpretation for nonstandard arithmetic*, Ann. Pure Appl. Logic **163** (2012), 1962–1994.
- [2] Damir D. Dzhalalov, *Reverse Mathematics Zoo*. <http://rmzoo.uconn.edu/>.
- [3] Albert E. Hurd and Peter A. Loeb, *An introduction to nonstandard real analysis*, Pure and Applied Mathematics, vol. 118, Academic Press Inc., Orlando, FL, 1985.
- [4] Ulrich Kohlenbach, *Applied proof theory: proof interpretations and their use in mathematics*, Springer Monographs in Mathematics, Springer-Verlag, Berlin, 2008.
- [5] ———, *Higher order reverse mathematics*, Reverse mathematics 2001, Lect. Notes Log., vol. 21, ASL, 2005, pp. 281–295.
- [6] Edward Nelson, *Internal set theory: a new approach to nonstandard analysis*, Bull. Amer. Math. Soc. **83** (1977), no. 6, 1165–1198.
- [7] Abraham Robinson, *Non-standard analysis*, North-Holland, Amsterdam, 1966.
- [8] Sam Sanders, *The taming of the Reverse Mathematics zoo*, Submitted, <http://arxiv.org/abs/1412.2022> (2015).
- [9] ———, *The refining of the taming of the Reverse Mathematics zoo*, To appear in Notre Dame Journal for Formal Logic, <http://arxiv.org/abs/1602.02270> (2016).
- [10] ———, *The unreasonable effectiveness of Nonstandard Analysis*, Submitted, <http://arxiv.org/abs/1508.07434> (2015).
- [11] Stephen G. Simpson (ed.), *Reverse mathematics 2001*, Lecture Notes in Logic, vol. 21, ASL, La Jolla, CA, 2005.
- [12] ———, *Subsystems of second order arithmetic*, Perspectives in Logic, CUP, 2009.
- [13] Stephen G. Simpson and Keita Yokoyama, *A nonstandard counterpart of WWKL*, Notre Dame J. Form. Log. **52** (2011), no. 3, 229–243.
- [14] Manfred Wolff and Peter A. Loeb (eds.), *Nonstandard analysis for the working mathematician*, Mathematics and its Applications, vol. 510, Kluwer Academic Publishers, 2000.
- [15] Xiaokang Yu, *Lebesgue convergence theorems and reverse mathematics*, Math. Logic Quart. **40** (1994), no. 1, 1–13.
- [16] Xiaokang Yu and Stephen G. Simpson, *Measure theory and weak König’s lemma*, Arch. Math. Logic **30** (1990), no. 3, 171–180.